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A Note on Superdistributions and Wavefront Set

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We present a simple and new method of constructing superdistributions on superspace over a Grassmann-Banach algebra, which close to the de Rham's "currents" defined as dual objects to differential forms. The paper also contains the extension of the Hörmander's description of the singularity structure (wavefront set) of a distribution to include the supersymmetric case.

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1. INTRODUCTION

In this work, we extend the definition of the objects most widely used in physics: distributions. The distribution theory is a generalization of the classical analysis, which makes it possible to deal in a systematic way difficulties as the unpleasant fact that not every function is differentiate. As a matter of fact, the space of distributions is essentially the smallest extension of the space of continuous functions where differentiation is always well defined. The theory was intensively developed by many mathematicians and theoretical physicists, mainly in connection with the needs of theoretical and mathematical physics. It one relies fundamentally on the notion of topological vector spaces. The Quantum Field Theory (QFT) perhaps is the example more important where technical results from distribution theory are required. In particular, Schwartz's theory of tempered distributions became fundamental to the Gårding-Wigthman axiomatization of relativistic QFT (Streater and Wightman, 1989; Bogoliubov *et al.*, 1990). In the same way the Fourier analysis of distributions plays an important role in the QFT, mainly in the spectral analysis of singularities. With this paper, we intend to define superdistributions and Fourier transformations in the supersymmetric field theory in the spirit of Schwartz's distributions and in the spirit of Hörmander's spectral analysis of singularities of distributions.

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Over the last decades, supersymmetric quantum theories have been studied intensively with the belief that such theories may play a part in a unified theory of fundamental forces, and many issues are understood much better now. These theories are usually characterized by their invariance properties with respect to transformations that involve anticommuting parameters. The latter play an essential role in the formulation of supersymmetric theories and their use sometimes facilities calculations, for instance in perturbation theory. As it occurs with the ordinary quantum field theories, supersymmetric field theories are also deeply connected to the presence of ultraviolet divergences, in a naive approach. However, physicists have soon learned how to make sense out of them in a mathematically proper way through the procedure now known as renormalization (a comprehensive account of the quantum theory through the algebraic renormalization approach can be found in the textbook by Piguet and Sibold (1986)). As first indicated by Wess and Zumino, supersymmetry is preserved by renormalization and further leads to a less divergent than conventional field theoretic model.

It is already well-known that the singularity structure of Feynman (or more precisely Wightman) superfunctions is completely associated with the "bosonic" sector of the superspace - the body of superspace. This result can be mainly justified by the heuristic form of defining superspace and superfields. It is, therefore, a natural question to ask how a mathematically rigorous definition of the structure of these singularities can be given. Although claims exist that such a result is completely obvious, we do not think that a clear proof is available in the published literature, to the best of our knowledge. However, to our great surprise, such a proof does exist and is extremely simple. The key ingredients in our analysis are the notion of the wavefront set (Hörmander, 1990, 1971; Duistermaat and Hörmander, 1972) of a superdistribution and the appropriate construction of Rogers of a superspace and superfields (Rogers, 1980). The notion of wavefront set was introduced by the mathematicians Hörmander and Duistermaat (Hörmander, 1971; Duistermaat and Hörmander, 1972) in the seventies and it is growing of importance, with a range of applications going beyond the original problems of linear partial equations. It has received, in the last years, a lot of attention from community of theoretical physicists in order to solve some important problems, such as the characterization of the spectral condition for a QFT on a general manifold (Radzikowski, 1992; Brunetti *et al.*, 1996).

This note is organized as follow: in Section 2, for the convenience of readers, we shall briefly review some few basic properties of superspaces based on the Rogers' work (Rogers, 1980). In Section 3, a new formulation of superdistributions on superspace is presented.[2] Such a formulation close to the de Rham's "currents" defined as dual objects to differential forms (De Rham, 1960). In Section 4, we

[2] An alternative formulation of superdistributions is given in the Ref. (Nagamachi and Kobayashi, 1988).

extend the notion of the wavefront set of a superdistribution. The well-known result that the singularities of a superdistribution may be expressed in a very simple way through the ordinary distribution is proved by functional analytical methods. Section 5 contains our conclusions. Finally, for sake of completeness, in the Appendix A we recall some properties of the microlocal analysis.

2. NOTIONS OF SUPERSPACE

This section introduces some few basic fundamentals on the theory of superspace. We follow here the work of Rogers (Rogers, 1980). Rogers' theory has an advantage, a superspace is an ordinary Banach manifold endowed with a Grassmann algebra structure, so that the topological constructions have their standard meanings.

We start by introducing first some definitions and concepts of a Grassmann-Banach algebra, i.e., a Grassmann algebra endowed with a Banach algebra structure. Let L be a finite positive integer. Denote by $\mathcal{G}$ a Grassmann algebra, such that $\mathcal{G}$ can naturally be decomposed as the direct sum $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$, where $\mathcal{G}_0$ consists of the even (commuting) elements and $\mathcal{G}_1$ consists of the odd (anti-commuting) elements in $\mathcal{G}$, respectively. Let M_L denote the set of sequences $\{(\mu_1, \ldots, \mu_k)| 1 \leq k \leq L; \mu_i \in \mathbb{N}; 1 \leq \mu_1 < \cdots < \mu_k \leq L\}$. Let Ω represent the empty sequence in M_L, and (j) denote the sequence with just one element j. A basis of $\mathcal{G}$ is given by monomials of the form $\{\xi_\Omega, \xi^{\mu_1}\xi^{\mu_2}, \ldots, \xi^{\mu_1}\xi^{\mu_2}\cdots\xi^{\mu_k}\}$ for all $\mu \in M_L$, such that $\xi_\Omega = \mathbb{I}$ and $\xi^{(i)}\xi^{(j)} + \xi^{(j)}\xi^{(i)} = 0$ for $1 \leq i, j \leq L$. Futhermore, there is no other independent relations among the generators. By $\mathcal{G}_L$ we denote the Grassmann algebra with L generators, where the even and the odd elements, respectively, take their values. L being assumed a finite integer (the number of generators L could be possibly infinite), it means that the sequence terminates at $\xi^1 \ldots \xi^L$ and there are only 2^L distinct basis elements. An arbitrary element $q \in \mathcal{G}_L$ has the form

$$q = q_{\mathrm{b}} + \sum_{(\mu_1,\ldots,\mu_k) \in M_L} q_{\mu_1,\ldots,\mu_k}\xi^{\mu_1}\ldots\xi^{\mu_k}, \tag{2.1}$$

where $q_{\mathrm{b}}, q_{\mu_1\ldots\mu_k}$ are real numbers. An even or odd element is specified by 2^{L-1} real parameters. The number q_{b} is called the body of q, while the remainder $q - q_{\mathrm{b}}$ is the soul of q, denoted $s(q)$. The element q is invertible if, and only if, its body is non-zero.

With reference to supersymmetric field theories, the commuting variable x has the form

$$x = x_{\mathrm{b}} + x_{ij}\xi^i\xi^j + x_{ijkl}\xi^i\xi^j\xi^k\xi^l + \cdots, \tag{2.2}$$

where $x_{\mathrm{b}}, x_{ij}, x_{ijkl}, \ldots$ are real variables. Similarly, the anticommuting variables (in the Weyl representation) θ and $\bar{\theta} = (\theta)^*$ have the form

$$\theta = \theta_i \xi^i + \theta_{ijk} \xi^i \xi^j \xi^k + \cdots, \qquad \bar{\theta} = \bar{\theta}_i \xi^i + \bar{\theta}_{ijk} \xi^i \xi^j \xi^k + \cdots, \tag{2.3}$$

where $\theta_i, \theta_{ijk}, \ldots$ are complex variables. The summation over repeated indices is to be understood unless otherwise stated. As pointed out by Vladimirov and Volovich (1984), from the physical point of view, superfields are not functions of $\theta_i, \theta_{ijk}, \ldots$ and $x_{\mathrm{b}}, x_{ij}, x_{ijkl}, \ldots$, but only depend on these variables through θ and x, as it occurs with ordinary complex analysis where analytic functions of the complex variables $z = x + iy$ are not arbitrary functions of the variables x and y, but functions that depend on x and y through z.

The Grassmann algebra may be topologized. Consider the complete norm on $\mathcal{G}_L$ defined by Rudolph (2000):

$$\|q\|_p = \left(|q_{\mathrm{b}}|^p + \sum_{(\mu)=1}^{L} |q_{\mu_1 \ldots \mu_k}|^p \right)^{1/p}. \tag{2.4}$$

A useful topology on $\mathcal{G}$ is the topology induced by this norm. The norm $\|\cdot\|_1$ is called the Rogers norm and $\mathcal{G}_L(1)$ the Rogers algebra (Rogers, 1980). The Grassmann algebra $\mathcal{G}$ equipped with the norm (2.4) becomes a Banach space. In fact $\mathcal{G}$ becomes a Banach algebra, i.e., $\|\mathbb{I}\| = 1$ and $\|qq'\| \leq \|q\|\|q'\|$ for all $q, q' \in \mathcal{G}$.

Definition 2.1. A Grassmann-Banach algebra is a Grassmann algebra endowed with a Banach algebra structure.

A superspace must be constructed using as a building block a Grassmann-Banach algebra $\mathcal{G}_L$ and not only a Grassmann algebra.

Definition 2.2. Let $\mathcal{G}_L = \mathcal{G}_{L,0} \oplus \mathcal{G}_{L,1}$ be a Grassmann-Banach algebra. Then the (m, n)-dimensional superspace is the topological space $\mathcal{G}_L^{m,n} = \mathcal{G}_{L,0}^m \times \mathcal{G}_{L,1}^n$ which generalizes the space $\mathbb{R}^m$, consisting of the Cartesian product of m copies of the even part of $\mathcal{G}_L$ and n copies of the odd part.

In supersymmetric quantum field theory, superfields are functions in superspace usually given by their (terminating) standard expansions in powers of the odd coordinates

$$F(x, \theta, \bar{\theta}) = \sum_{(\gamma)=0}^{\Gamma} f_{(\gamma)}(x)(\theta)^{(\gamma)}, \tag{2.5}$$

where $(\theta)^{(\gamma)}$ comprises all monomials in the anticommuting variables θ and $\bar{\theta}$ (belonging to odd part of a Grassmann-Banach algebra) of degree $|\gamma|$; $f_{(\gamma)}(x)$ is called a component field, whose Lorentz properties are determined by those of

$F(x, \theta, \bar{\theta})$ and by the power (γ) of (θ). The following notation, extended to more than one θ variable, is used (2.5): $(\theta) = (\theta_1, \bar{\theta}_1, \ldots, \theta_n, \bar{\theta}_n)$, and (γ) is a multi-index $(\gamma_1, \bar{\gamma}_1, \ldots, \gamma_n, \bar{\gamma}_n)$ with $|\gamma| = \sum_{r=1}^{n}(\gamma_r + \bar{\gamma}_r)$ and $(\theta)^{(\gamma)} = \Pi_{r=1}^{m} \theta_r^{\gamma_r} \bar{\theta}_r^{\bar{\gamma}_r}$. In Eq. (2.5), for a (4,4)-dimensional superspace, $\Gamma = (2, 2)$.

Rogers (Rogers, 1980) considered superfields in $\mathcal{G}_L^{m,n}$ as G^∞ superfunctions, i.e., functions whose coefficients $f_{(\gamma)}(x)$ of their expansions are smooth functions of $\mathbb{R}^m$ into $\mathcal{G}_L$, extended from $\mathbb{R}^m$ to all of $\mathcal{G}_L^{m,0}$ by z-continuation. Throughout the remainder of this paper the prefix "super" is used for entities involving odd Grassmann variables.

Definition 2.3. Let U be an open set in $\mathcal{G}_L^{m,0}$ and $\epsilon : \mathcal{G}_L^{m,0} \to \mathbb{R}^m$ be the body projection which associates to each m-tuple $(x_1, \ldots, x_m)$ in $\mathcal{G}_L^{m,0}$ the m-tuple $(\epsilon(x_1), \ldots, \epsilon(x_m))$ in $\mathbb{R}^m$. Let V be an open set in $\mathbb{R}^m$ with $V = \epsilon(U)$. We get through z-continuation – or "Grassmann analytic continuation" – of a function $f \in C^\infty(V, \mathcal{G}_L)$ a function $z(f) \in G^\infty(U, \mathcal{G}_L)$, which admits an expansion in powers of the soul of x

$$z(f)(x_1, \ldots, x_m) = \sum_{i_1=\cdots=i_m=0}^{L} \frac{1}{i_1! \cdots i_m!} \left[\partial_1^{i_1} \cdots \partial_m^{i_m}\right] f(\epsilon(x)) s(x_1)^{i_1} \cdots s(x_m)^{i_m},$$

where $s(x_i) = (x_i - \epsilon(x_i))$ and $\epsilon(x_i) = (x_i)_b$.

One should keep always in mind that the continuation involves only the even variables $z : C^\infty(\epsilon(U)) \to G^\infty(U)$, and that $z(f)(x_1, \ldots, x_m)$ is a supersmooth function if their components are smooth for soulless values of x. This justifies the formal manipulations in the physics literature, where superfields are manipulated as if their even arguments were ordinary numbers (Rabin, 1985, 1986): a supersmooth function is completely determined when its components are known on the body of superspace.

According to Definition 2.3, the superfield $F(x, \theta, \bar{\theta}) \in G^\infty(U, \mathcal{G}_L)$ admits an expansion

$$F(x, \theta, \bar{\theta}) = \sum_{(\gamma)=0}^{\Gamma} z(f_{(\gamma)})(x)(\theta)^{(\gamma)},$$

but here with suitable $f_{(\gamma)} \in C^\infty(\epsilon(U), \mathcal{G}_L)$.

3. DISTRIBUTIONS ON THE SUPERSPACE

We begin by introducing the concept of superdistributions as the dual space of supersmooth functions in $\mathcal{G}_L^{m,0}$, with compact support, equipped with an appropriate topology, called *test superfunctions*. This can be done relatively straightforward in analogy to the notion of distributions as the dual space to the space $C_0^\infty(U)$ of functions on an open set $U \subset \mathbb{R}^m$ which have compact support, since the spaces $\mathcal{G}_L^{m,0}$ and $\mathcal{G}_L^{m,n}$ are regarded as ordinary vector spaces of $2^{L-1}(m)$ and $2^{L-1}(m+n)$ dimensions, respectively, over the real numbers.

Let $\Omega \subset \mathbb{R}^m$ be an open set. $\Omega = \epsilon(U)$ regarded as a subset of $\mathcal{G}_L^{m,0}$, it is identified with the body of some domain in superspace. Let $C_0^\infty(\Omega, \mathcal{G}_L)$ be the space of $\mathcal{G}_L$-valued smooth functions with compact support in $\mathcal{G}_L$. Every function $f \in C_0^\infty(\Omega, \mathcal{G}_L)$ can be expanded in terms of the basis elements of $\mathcal{G}_L$ as:

$$f(x) = \sum_{(\mu_1,\ldots,\mu_k)\in M_L^0} f_{\mu_1,\ldots,\mu_k}(x)\xi^{\mu_1}\cdots\xi^{\mu_k}, \tag{3.1}$$

where $M_L^0 \stackrel{\text{def}}{=} \{(\mu_1,\ldots,\mu_k)|0 \le k \le L; \mu_i \in \mathbb{N}; 1 \le \mu_1 < \cdots < \mu_k \le L\}$ and $f_{\mu_1,\ldots,\mu_k}(x)$ is in the space $C_0^\infty(\Omega)$ of real-valued smooth functions on Ω with compact support. Thus, it follows that the space $C_0^\infty(\Omega, \mathcal{G}_L)$ is isomorphic to the space $C_0^\infty(\Omega) \otimes \mathcal{G}_L$ (Nagamachi and Kobayashi, 1988). In accordance with the Definition 2.3, the smooth functions of $C_0^\infty(\Omega, \mathcal{G}_L)$ can be extended from $\Omega \subset \mathbb{R}^m$ to $U \subset \mathcal{G}_L^{m,0}$ by Taylor expansion.

In order to define superdistributions, we need to give a suitable topological structure to the space $G_0^\infty(U, \mathcal{G}_L)$ of $\mathcal{G}_L$-valued superfunctions on an open set $U \subset \mathcal{G}_L^{m,0}$ which have compact support. According to a proposition by Rogers, every G^∞ superfunction on a compact set $U \subset \mathcal{G}_L^{m,0}$ can be considered as a real-valued C^∞ function on $U \subset \mathbb{R}^N$, where $N = 2^{L-1}(m)$, regarding $\mathcal{G}_L^{m,0}$ and $\mathcal{G}_L$ as Banach spaces. In fact, the identification of $\mathcal{G}_L^{m,0}$ with $\mathbb{R}^{2^{L-1}(m)}$ is possible (Catenacci *et al.*, 1985). We have here an example of functoriality. Indeed, let X and Y denote a G^∞ supermanifold and a Banach manifold C^∞ , respectively. Then with each supermanifold X we associate a Banach manifold Y, via a *covariant* functorial relation $\lambda : X \to Y$, and with each G^∞ map ϕ defined on X, a C^∞ map $\lambda(\phi)$ defined on Y (Catenacci *et al.*, 1985).

Following, we shall first consider only the subset C_K^∞ of $C_0^\infty(U \subset \mathbb{R}^N)$ which consists of functions with support in a fixed compact set K. Since by construction C_K^∞ is a Banach space, the functions C_K^∞ have a natural topology given by the finite family of norms

$$\|\phi\|_{K,m} = \sup_{\substack{|p|\le m\\ x\in K}} |D^p\phi(x)|, \quad D^p = \frac{\partial^{|p|}}{\partial x_1^{p_1}\cdots\partial x_m^{p_m}}, \tag{3.2}$$

where $p = (p_1, p_2, \ldots, p_m)$ is a m-tuple of non-negative integers, and $|p| = p_1 + p_2 + \ldots + p_m$ defines the order of the derivative. Next, let U be considered as a union of compact sets K_i which form an increasing family $\{K_i\}_{i=1}^\infty$, such that K_i is contained in the interior of K_{i+1}. That such family exist follows from the Lemma 10.1 of (Treves, 1967). Therefore, we think of $C_0^\infty(U \subset \mathbb{R}^N)$ as $\cup_i C_{K_i}^\infty(U \subset \mathbb{R}^N)$. We take the topology of $C_0^\infty(U \subset \mathbb{R}^N)$ to be given by the strict inductive limit topology of the sequence $\{C^\infty_{K_i}(U \subset \mathbb{R}^N)\}$. Of another way, we may define convergence in $C^\infty_0(U \subset \mathbb{R}^N)$ of a sequence of functions $\{\phi_k\}$ to mean that for each k, one has supp $\phi_k \subset K \subset U \subset \mathbb{R}^N$ such that for a function $\phi \in C^\infty_0(U \subset \mathbb{R}^N)$ we have $\|\phi - \phi_k\|_{K,m} \to 0$ as $k \to \infty$. This notion of convergence generates a topology which makes $C^\infty_0(U \subset \mathbb{R}^N)$, certainly, a topological vector space.

Now, let F and E be spaces of smooth functions with compact support defined on $U \subset \mathcal{G}^{m,0}_L$ and $U \subset \mathbb{R}^N$, respectively. If $\lambda : \mathrm{E} \to \mathrm{F}$ is a *contravariant* functor which associates with each smooth function of compact support in E, a smooth function of compact support in F, then we have a map

$$\|\phi\|_{K,m} \longrightarrow \|\lambda(\phi)\|_{K,m}, \tag{3.3}$$

providing $G^\infty_0(U, \mathcal{G}_L)$ with a limit topology induced by a finite family of norms.

We now take a result by Jadczyk and Pilch (1981), later refined by Hoyos *et al.* (1984), which establishes as a natural domain of definition for supersmooth functions a set of the form $\epsilon^{-1}(\Omega)$, where Ω is open in $\mathbb{R}^m$. Let $\epsilon^{-1}(\Omega)$ be the domain of definition for a superfunction $f \in G^\infty_0(\epsilon^{-1}(\Omega), \mathcal{G}_L)$, where $\epsilon^{-1}(\Omega)$ is an open subset in $\mathcal{G}^{m,0}_L$ and Ω is an open subset in $\mathbb{R}^m$, and let $\tilde{\phi} \in C^\infty_0(\Omega, \mathcal{G}_L)$ denotes the restriction of ϕ to $\Omega \subset \mathbb{R}^m \subset \mathcal{G}^{m,0}_L$. Then, it follows that $(\partial^{p_1}_1 \cdots \partial^{p_m}_m \phi)^\sim = \partial^{p_1}_1 \cdots \partial^{p_m}_m \tilde{\phi}$, where the derivatives on the right-hand side are with respect to m real variables. Now, suppose $\Omega = \cup_i \tilde{K}_i$ where each $\tilde{K}_i$ is open and has compact closure in $\tilde{K}_{i+1}$. It follows that $C^\infty_0(\Omega, \mathcal{G}_L) = \cup_i C^\infty_{\tilde{K}_i}(\Omega, \mathcal{G}_L)$. Then, one can give $C^\infty_0(\Omega, \mathcal{G}_L)$ a limit topology induced by finite family of norms (Nagamachi and Kobayashi, 1988)

$$\|\tilde{\phi}\|_{\tilde{K},m} = \sup_{\substack{|p|\le m \\ x\in\tilde{K}}} |D^p\tilde{\phi}(x)| = \sup_{\substack{|p|\le m \\ x\in\tilde{K}}} \left\{ \sum_{(\mu_1,\ldots,\mu_k)\in M^0_L} |D^p\tilde{\phi}_{\mu_1,\ldots,\mu_k}(x)| \right\}. \tag{3.4}$$

Finally, a suitable topological structure to the space $G^\infty_0(U, \mathcal{G}_L)$ of $\mathcal{G}_L$-valued superfunctions on an open set $U \subset \mathcal{G}^{m,n}_L$ which have compact support, it is obtained immediately by the natural identification of $\mathcal{G}^{m,n}_L$ with $\mathbb{R}^{2^{L-1}(m+n)}$ and by the obvious extension of the construction above, which allows us define a limit topology induced to the space $G^\infty_0(U, \mathcal{G}_L)$ by finite family of norms,

$$\|\lambda(\phi)\|_{K,m+n} = \sup_{\substack{|p|\le m+n \\ z\in K}} |D^p(\lambda(\phi))(z)|, \quad D^p = \frac{\partial^{|q|+|r|}}{\partial x^{q_1}_1 \cdots \partial x^{q_m}_m \partial\theta^{r_1}_1 \cdots \partial\theta^{r_n}_n} \tag{3.5}$$

where the derivatives $\partial^{|q|}/\partial x^{q_1}_1 \cdots \partial x^{q_m}_m$ commute while the derivatives $\partial^{|q|}/\partial\theta^{r_1}_1 \cdots \partial\theta^{r_n}_n$ anti-commute, and $|p| = |q| + |r| = \sum^m_{i=1} q_i + \sum^n_{j=1} r_j$ defines the total order of the derivative, with $r_j = 0, 1$.

We are now ready to define a superdistribution in an open subset U of $\mathcal{G}^{m,n}_L$. The set of all superdistributions in U will be denoted by $\mathfrak{D}'(U)$. A superdistribution is a continuous linear functional $u : G_0^\infty(U) \to \mathcal{G}_L$, where $G_0^\infty(U)$ denotes the test superfunction space of $G^\infty(U)$ superfunctions with compact support in $K \subset U$. The continuity of u on $G_0^\infty(U)$ is equivalent to its boundedness on a neighbourhood of zero, i.e., the set of numbers $u(\phi)$ is bounded for all $\phi \in G_0^\infty(U)$. The last statement translates directly into:

Propoition 3.1. *A superdistribution u in $U \in \mathcal{G}_L^{m,n}$ is a continuous linear functional on $G_0^\infty(U)$ if and only if to every compact set $K \subset U$, there exists a constant C and $(m+n)$ such that*

$$|u(\phi)| \leq C \sup_{\substack{|p|\leq m+n \\ z\in K}} |D^p(\phi)(z)|, \quad \phi \in G_0^\infty(K).$$

Proof: See Ref. Franco and Polito (2004) □

4. WAVEFRONT SET OF A SUPERDISTRIBUTION

A great deal of progress has been made in recent years in characterizing the "ultraviolet divergences" of quantum fields in curved spacetime and developing renormalization theory for interacting quantum fields by the use of the methods of "microlocal analysis." This leads to the definition of the wavefront set, denoted (WF), of a distribution, a refined description of the singularity spectrum. Similar notion was developed in other versions by (Sato, 1969), (Iagolnitzer, 1975) and (Sjöstrand, 1982). The definition as known nowadays is due to Hörmander. He used this terminology due to an existing analogy between his studies on the "propagation" of singularities and the classical construction of propagating waves by Huyghens. For a distribution u we introduce its wavefront set $WF(u)$ as a subset in phase space $\mathbb{R}^n \times \mathbb{R}^n$.[3] We shall be thinking of points (x, k) in phase space as specifying those singular directions k of a "bad" behaviour of the Fourier transform $\hat{u}$ at infinity that are responsible for the non-smoothness of u at the point x in position space. So we shall usually want $k \neq 0$. A relevant point is that $W\ F(u)$ is independent of the coordinate system chosen, and it can be described locally.

It is well-known that the regularity properties of a distribution are in correspondence with the decay properties of its Fourier transform (see Appendix A for details). The results which now follow prove that the decay properties of a superdistribution at infinity and the smoothness properties of its Fourier transform are analogous to the case of ordinary distributions, i.e., no new singularity appear by taking into account the structure of the superspace.

[3] The functorially correct definition of phase space is $\mathbb{R}^n \times (\mathbb{R}^n)^*$. We shall here ignore any attempt to distinguish between $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$.

Lemma 4.1. *Let $X \subset \mathcal{G}_L^{m,0}$ be an open set, and u be a superdistribution on X taking values in $\mathcal{G}_L$, i.e., a linear functional $u : G_0^\infty(X) \to \mathcal{G}_L$. Let ϕ be a supersmooth function with compact support $K \subset X$. Then ϕu is also supersmooth on K, if its components $(\phi u)(\epsilon(x))$ are smooth on a compact set $K' \subset \Omega$, where Ω is the body of superspace. Therefore, the following estimate holds:*

$$|\widehat{\phi u}(k)| \leq (1 + |k_{\mathrm{b}}|)^{-N} C(N, \phi).$$

Proof: See Ref. Franco and Polito (2004) □

Lemma 4.2. *By replacing $\mathcal{G}_L^{m,0}$ by $\mathcal{G}_L^{m,n}$ in the Lemma 4.1, then the following estimate holds:*

$$|\widehat{\phi u}(k, \theta, \bar{\theta})| \leq (1 + |k_{\mathrm{b}}|)^{-N} C(N, \phi_{(\gamma)}) \|\theta_1\| \|\bar{\theta}_1\| \cdots \|\theta_n\| \|\bar{\theta}_n\|.$$

Proof: See ref. Franco and Polito (2004) □

Combining the results above, we have proved:

Theorem 4.3. *The singularities of a superdistribution u are located at specific values of the body of x, the coordinates of the physical spacetime, independently of the odd coordinates.*

We sum up the preceding discussion as follows:

Definition 4.4. (Wavefront Set of a Superdistribution). The wavefront set $WF(u)$ of a superdistribution u in a superspace $\mathcal{M}$ is the complement of the set of all regular directed points in the cotangent bundle $T^*\mathcal{M}_0$, where $\mathcal{M}_0 = \epsilon(\mathcal{M})$ is the body of superspace, excluding the trivial point $k_{\mathrm{b}} = 0$.

Remark 4.1. *A direction k_{b} for which the Fourier transform of a superdistribution u shows to be of fast decrease is called to be a regular direction of $\hat{u}$. Therefore, in order to determine whether $(x_{\mathrm{b}}, k_{\mathrm{b}})$ belongs to the wavefront set of u one must first to localize u around x_{b}, next to obtain Fourier transform $\hat{u}$ and finally to look at the decay in the direction k_{b}. Hence, the wavefront set not only describes the set of points where a superdistribution is singular, but it also localizes the frequencies that constitute these singularities.*

There is a more precise version of Definition 4.4. As we have seen in Section 3 all of the foregoing definitions and statements about supermanifolds may be converted into corresponding definitions and statements about ordinary manifolds, since associated with a supermanifold $\mathcal{M}$ of dimension (m, n) is a family of ordinary manifolds, of dimensions $N = 2^{L-1}(m+n)$, $(L = 1, 2, \ldots)$. The resulting manifold is called the Lth skeleton of $\mathcal{M}$ and denoted by $\mathcal{S}_L(\mathcal{M})$ (DeWitt, 1992). With the aid of the family of skeletons we can define the pushforward (or direct image) of a superdistribution. Let $X \subset \mathcal{S}_L(\mathcal{M})$ and $Y \subset \mathcal{M}_0$ be open sets and let ϵ be the natural projection from $\mathcal{S}_L(\mathcal{M})$ (or $\mathcal{M}$) to $\mathcal{M}_0$, the body map. If we introduce local coordinates $x = (x_1, \ldots, x_N)$ in X, then Y is defined by $x_{\mathrm{b}} = (x_1, \ldots, x_m)$. There is a local relationship between the body and the skeletons given by

$$\mathcal{S}_L(X) \overset{\text{diff.}}{=} Y \times \mathbb{R}^{2^{L-1}(m+n)-m}.$$

Now, let u be a superdistribution on X, then the pushforward $\epsilon_* u$ defined by $\epsilon_* u(\varphi) = u(\epsilon^* \varphi)$, $\varphi \in C_0^\infty(Y)$, it is a superdistribution on Y. Using these concepts, we can establish the following

Corollary 4.5. *Let $\epsilon : X \subset \mathcal{S}_L(\mathcal{M}) \to Y \subset \mathcal{M}_0$ be the body projection, and let $u \in \mathfrak{D}'(X)$. Then*

$WF(\epsilon_* u) \subset \{(x_{\mathrm{b}}, k_{\mathrm{b}}) \in T^*\mathcal{M}_0 \backslash 0 | \exists\, x' = (x_{m+1}, \ldots, x_{N'}), (x_{\mathrm{b}}, x', k_{\mathrm{b}}, 0) \in WF(u)\}$ *where* $N' = 2^{L-1}(m+n) - m$.

Proof: See Ref. Franco and Polito (2004) □

5. CONCLUSIONS

We have introduced a notion of superdistribution in superspace which seems to have some advantages: by exploring the functorial relations between a G^∞-superspace and a family of Banach manifolds C^∞ we define the space of superdistributions as the dual of the test function space of C^∞-functions with compact support endowed with a suitable topology on Banach spaces. In particular, Wightman superfunctions and superpropagators, which appear in the supersymmetric quantum field theory, can be treated as our superdistributions. We have also obtained useful results on the singularity structure of such objects, here analysed in the context of the development of the potent mathematical tool of microlocal analysis and characterized in terms of the its wavefront set. Our analysis represents only the first step towards a more interesting physically situation: the perturbative treatment of interacting quantum superfield models, in particular the formulation of renormalization theory on curved supermanifolds.

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APPENDIX. MICROLOCAL ANALYSIS: REVIEW OF SOME BASIC IDEAS

In this appendix we briefly recall some standard facts on microlocal analysis. The key point of the microlocal analysis is the transference of the study of singularities of distributions from the configuration space only to the rather phase space, by exploring in frequency space the decay properties of a distribution at infinity and the smoothness properties of its Fourier transform. As it is well-known (Hörmander, 1990; Reed and Simon, 1975), a distribution of compact support, $u \in \mathcal{E}'(\mathbb{R}^n)$, is a smooth function if, and only if, its Fourier transform, $\widehat{u}$, rapidly decreases at infinity (i.e., as long as supp u does not touch the singularity points). By a fast decay at infinity, one must understanding that for all positive integer N exists a constant C_N, which depends on N, such that

$$|\widehat{u}(k)| \leq (1 + |k|)^{-N} C_N, \quad \forall N \in \mathbb{N};\ k \in \mathbb{R}^n. \tag{A.1}$$

If, however, $u \in \mathcal{E}'(\mathbb{R}^n)$ is not smooth, then the directions along which $\widehat{u}$ does not fall off sufficiently fast may be adopted to characterize the singularities of u.

For a distributions does not necessarily of compact support, still we can verify if its Fourier transform rapidly decreases in a given region V through the technique of localization. More precisely, if $V \subset X \subset \mathbb{R}^n$ and $u \in \mathcal{D}'(X)$, we can restrict u to a distribution $u|_V$ in V by setting $u|_V(\phi) = u(\phi)$, where ϕ is a smooth function with support contained in a region V. The distribution ϕu can then be seen as a distribution of compact support on $\mathbb{R}^n$. Its Fourier transform will be defined as a distribution on $\mathbb{R}^n$, and must satisfy, in absence of singularities in $V \subset \mathbb{R}^n$, the property (A.1). From this point of view, all development is local in the sense that only the behaviour of the distribution on the arbitrarily small neighbourhood of the singular point, in the configuration space, is relevant.

Let $u \in \mathcal{D}'(\mathbb{R}^n)$ be a distribution and $\phi \in C_0^\infty(V)$ a smooth function with support $V \subset \mathbb{R}^n$. Then, ϕu has compact support. The Fourier transform of ϕu produces a smooth function in frequency space.

Lemma A.1. *Consider $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\phi \in C_0^\infty(V)$. Then $\widehat{\phi u}(k) = u(\phi e^{-ikx})$. Moreover, the restriction of u to $V \subset \mathbb{R}^n$ is smooth on V if, and only if, for every $\phi \in C_0^\infty(V)$ and each positive integer N there exist a constant $C(\phi, N)$, which depends on N and ϕ, such that $|\widehat{\phi u}(k)| \leq (1 + |k|)^{-N} C(\phi, N)$ for all $N \in \mathbb{R}$ and $k \in \mathbb{R}^n$.*

If $u \in \mathcal{D}'(\mathbb{R}^n)$ is singular in x, and $\phi \in C_0^\infty(V)$ is $\phi(x) \neq 0$; then ϕu is also singular in x and has compact support. However, in some directions in k-space

$\widehat{\phi u}$ until will be asymptotically limited. This is called the set of *regular directions* of u.

Definition A.2. Let $u(x)$ be an arbitrary distribution, not necessarily of compact support, on an open set $X \subset \mathbb{R}^n$. Then, the set of pairs composed by singular points x in configuration space and by its associated nonzero singular directions k in Fourier space

$$WF(u) = \{(x, k) \in X \times (\mathbb{R}^n \backslash 0) | k \in \Sigma_x(u)\}, \tag{A.2}$$

is called *wavefront set* of $u.\Sigma_x(u)$ is defined to be the complement in $\mathbb{R}^n \backslash 0$ of the set of all $k \in \mathbb{R}^n \backslash 0$ for which there is an open conic neighbourhood M of k such that $\widehat{\phi u}$ rapidly decreases in M, for $|k| \to \infty$.

Remark. We will now collect some basic properties of the wavefront set:

(1) The $WF(u)$ is conic in the sense that it remains invariant under the action of dilatations, i.e., when we multiply the second variable by a positive scalar. This means that if $(x, k) \in WF(u)$ then $(x, \lambda k) \in WF(u)$ for all $\lambda > 0$.
(2) From the definition of $WF(u)$, it follows that $\pi_1(WF(u)) \to x$ is the projection onto the first variable, by consisting of those points that have no neighbourhood wherein u is a smooth function. The projection onto the second variable, $\pi_2(WF(u)) \to \Sigma_x(u)$, is the cone around k attached to a such point denoting the set of high-frequency directions responsible for the appearance of a singularity at this point.
(3) The wavefront set of a smooth function is the empty set.
(4) For all smooth function ϕ with compact suport $WF(\phi u) \subset WF(u)$.
(5) For any partial linear differential operator P, with C^∞ coefficients, we have

$$WF(Pu) \subseteq WF(u).$$

(6) If u and v are two distributions belonging to $\mathcal{D}'(\mathbb{R}^n)$, with wavefront sets $WF(u)$ and $WF(v)$, respectively; then the wavefront set of $(u + v) \in \mathcal{D}'(\mathbb{R}^n)$ is contained in $WF(u) \cup WF(v)$.
(7) If U, V are open set of $\mathbb{R}^n, u \in \mathcal{D}'(V)$, and $\chi : U \to V$ a diffeomorphism such that $\chi^* u \in \mathcal{D}'(U)$ is the distribution pulled back by χ, then $WF(\chi^* u) = \chi^* WF(u)$.

We emphasize that a number of operations, not possible in general, become feasible for distributions under special assumptions on their wavefront set, such as taking products. As a result of this, the wavefront set applies to theories which are formulated in terms of pointlike fields. In the naive perturbative scheme of quantum field theories, one encounters formal products of fields which are a priori ill-defined. This difficulty lies at the heart of renormalization theory. The latter starts from the observation that products of fields (operator-valued distributions) are well-defined on a subset which does not contain the diagonal (all coinciding points, or the zero section). Renormalization consists then in the continuation of products of distributions to the whole space.

In order to give precise statements to the product of these fields, we appeal to the criterion below.

Theorem A.3. (Hörmander's Criterion). *Let u and v be distributions; if the wavefront set of u and v are such that*

$$(x,0) \notin WF(u) \oplus WF(v) = \{(x, k_1 + k_2) | (x, k_1) \in WF(u),\ (x, k_2) \in WF(v)\},$$

then the product uv exists and $WF(uv) \subset WF(u) \cup WF(v) \cup (WF(u) \oplus WF(v))$.

Hence, the product of the distributions u and v is well-defined in x, if u, or v, or both distributions are regular in x. Otherwise, if u and v are singular in x, the product can still exist if, the sum of the second components of $WF(u)$ and $WF(v)$ related to x can be linearly combined to give zero only by a trivial solution.

Example. *The distributions $u, v \in \mathcal{D}'(\mathbb{R})$, $u(x) = \frac{1}{x+i\epsilon}$ and $v(x) = \frac{1}{x-i\epsilon}$, with the Heavyside distributions $\widehat{u}(k) = 2\pi i\theta(-k)$ and $\widehat{v}(k) = -2\pi i\theta(k)$ as their Fourier transforms, have the following wavefront sets:*

$$WF(u) = \{(0,k) | k \in \mathbb{R}^-\backslash 0\}, \quad WF(v) = \{(0,k) | k \in \mathbb{R}^+\backslash 0\}.$$

Thus, from the Hörmander's Criterion one finds that there exist the powers of u^n and v^n. On the other hand, the products between u and v do not match the above criterion and do not exist, indeed. The example clearly indicates that one can multiply distributions even if they have overlapping singularities, provided their wavefront sets are in favorable positions. Such an observation is significant because it makes clear that *the problem is not only where the support is, but in which directions the Fourier transform is not rapidly decreasing*!

Another result, which we merely state, is needed to complete this briefing on microlocal analysis.

Theorem A.4. (Wavefront set of pushforwards of a distribution). *Let $f : X \to Y$ be a submersion, and let $u \in \mathcal{E}'(X)$. Then*

$$WF(f_* u) \subset \{(f(x), \eta) | x \in X, (x, {}^t f'_x \eta) \in WF(u) \quad \text{or} \quad {}^t f'_x \eta = 0\},$$

where ${}^t f'_x$ denotes the transpose matrix of the Jacobian matrix f'_x of f.

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